Math 259A Lecture 24 Notes

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1 The Group Measure Space von Neumann Algebra Construction

1.1 Measure-preserving actions of groups

Up to now, our only examples of II_1 factors have been

- the hyperfinite II_1 factor R,
- the group von Neumann algebra $L(\Gamma)$, where Γ is ICC.

Here is another class of examples of II_1 factors.

Definition 1.1. Let Γ be a discrete, countable group, and let (X, μ) be a standard, nonatomic probability space. A **measure-preserving action** $\Gamma \circlearrowright^{\sigma} X$ is a collection of measure-preserving ($\mu(a\sigma_g^{-1}A) = \mu(A)$) maps $\sigma_g : X \to X$ that are invertible (mod null sets) such that $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$ sending $g \mapsto \sigma_g$ is a group homomorphism.

By $\mathrm{Aut}(X, \mu)$, we mean automorphisms of X as a measure space. A measure-preserving α gives rise to an map $\alpha^* \in \mathrm{Aut}(L^\infty(X, \mu), \int d\mu)$ given by $\alpha^*(f) = f \circ \alpha^{-1}$. The action $\Gamma \circlearrowright^{\sigma} (X, \mu)$ induces σ^*, an action of Γ on $(L^\infty(X, \mu), \int d\mu)$ by $(\sigma^*)_g(f) = f \circ (\sigma_g)^{-1}$; in particular, $(\sigma^*)_g(\sigma^*)_h = (\sigma^*)_{gh}$. That is, we gave a homomorphism $\sigma^* : \Gamma \to \mathrm{Aut}(L^\infty(X, \mu), \int \cdot\, d\mu)$, where this is the group of automorphisms of $L^\infty(X)$ preserving $\int \cdot\, d\mu$. We will denote this action by $\Gamma \circlearrowright^{\sigma} (L^\infty(X, \mu), \int \cdot\, d\mu)$, suppressing the star notation.

Surprisingly, we can go back!

Theorem 1.1 (von Neumann). *Let $\beta \in \mathrm{Aut}(L^\infty(X, \mu), \int \cdot\, d\mu)$. Then there exists a unique $\alpha \in \mathrm{Aut}(X, \mu)$ such that $\alpha^* = \beta$.*

1.2 Construction of the algebra

Here, $A = L^\infty$ is a von Neumann algebra. Form the vector space $A\Gamma$ of finitely supported sums $\sum_g a_g u_g : a_g \in A\}$. We can turn this into an algebra by introducing the multiplication

$$(a_g u_g) \cdot (a_h u_h) = a_g \sigma_g(a_h) u_{gh}.$$

This gives us an algebra where $u_g a_h u_g^{-1} = \sigma_g(a_h)$. Moreover, this is a *-algebra by

$$(a_g u_g)^* = u_{g^{-1}} \overline{a_g} = \sigma_{g^{-1}}(\overline{a_g}) u_{g^{-1}}.$$

We also have the functional

$$\tau\left(\sum a_g u_g\right) = \int a_e \, d\mu.$$

Proposition 1.1. *τ is a trace state.*

Proof. We want to show that $\tau((a_g u_g) \cdot (a_h u_h)) = \tau((a_h u_h) \cdot (a_g u_g))$. That is, we want to show that

$$\delta_{gh,e} \int a_g \sigma_g(a_h) \, d\mu = \delta_{hg,e} \int a_h \sigma_h(a_g) \ , d\mu.$$

Replacing $h = g^{-1}$, this is

$$\int a_g \sigma_g(a_{g^{-1}}) \, d\mu = \int a_h \sigma_h(a_{h^{-1}}) \, d\mu$$

If we apply $\sigma_{h^{-1}}$ to the right hand side, since $\sigma_{h^{-1}}$ preserves the integral, we get

$$\int \sigma_{h^{-1}}(a_h) a_{h^{-1}} \, d\mu.$$

So these are the same. $\square$

Now we can define $L^2(\Gamma \circlearrowright X)$, the completion of $(A\Gamma, \langle \cdot, \cdot \rangle_\tau)$. This completion is naturally isomorphic to $\ell^2(\Gamma, L^2(X))$. Alternatively, we can identifying it with the following: $\bigoplus_{g\in\Gamma}(L^2(X,\mu))_g$. We can also write it like $\{\sum_g \xi_g u_g : \xi_g \in L^2(X), \int \sum_{g\in\Gamma} |\xi_g|^2 \, d\mu < \infty\}$. That is, we want $\sum_{g\in\Gamma} \|\xi_g\|^2_{L^2(X)} < \infty$.

Example 1.1. If X is a single point, then $A = \mathbb{C}$. So this gives $\ell^2(\Gamma)$.

On $L^2(\Gamma \circlearrowright X)$, we define operators of left multiplication and right multiplication by elements in $A\Gamma$; this gives $x = \sum_g c_g u_g \mapsto \lambda(x)$ or $\rho(x)$. In particular, the operation is

$$\lambda(a_g u_g) \sum_h \xi_h u_h = \sum_h a_g u_g \xi_h u_h = \sum_h a_g \sigma_g(\xi_h) u_{gh} = \sum_{h'} a_g \sigma_g(a_{g^{-1}h'}) u_{h'}.$$

So we have two representations of the *-algebra $A\Gamma$ on $L^2(\Gamma \circlearrowright X)$. Check that $\lambda(x^*) = \lambda(x)^*$ and $\tau(x) = \left\langle \lambda(x)\hat{1}, \hat{1} \right\rangle_\tau$, where $\hat{1}$ is the series with the only nonzero coefficient u_1 and ξ_1 to be the constant 1 function.

Definition 1.2. The **group measure space construction** is the von Neumann algebra $L(\Gamma \circlearrowright X) := \overline{\lambda(A\Gamma)}^{\mathrm{wo}} \subseteq \mathcal{B}(L^2(\Gamma \circlearrowright X))$ (and similarly for R).

1.3 Properties of the algebra

We can extend τ to the whole space by $\tau(x) := \langle x(\hat{1}), \hat{1} \rangle$.

Theorem 1.2. *$(L(\Gamma \circlearrowleft X), \tau)$ and $(R(\Gamma \circlearrowleft X), \tau)$ are tracial von Neumann algebras (and thus finite) with the faithful, normal trace τ.*

Theorem 1.3. *The left and right group measure space constructions are each other's commutants. That is, $L(\Gamma \circlearrowleft X)' = R(\Gamma \circlearrowleft X)$ and $R(\Gamma \circlearrowleft X)' = L(\Gamma \circlearrowleft X)$.*

In fact, if $\xi = \sum_g \xi_g u_g \in L^2(\Gamma \circlearrowleft X)$ and η is similar, then we have the formal product $\xi\eta \in \ell^\infty(\Gamma, L^1(X))$. Then we call ξ a convolver if $\xi\eta \in L^2(\Gamma \circlearrowleft X)$ for all $\eta \in L^2(\Gamma \circlearrowleft X)$. Then we get a characterization of $L(\Gamma \circlearrowleft X)$ in terms of left multiplication by convolvers, just like in the $L(\Gamma)$ case.

Observe that $A = L^\infty(X, \mu)$ sits inside $L(\Gamma \circlearrowleft X)$ as the algebra $a \mapsto au_e$.

Theorem 1.4. *Let $\Gamma \circlearrowleft X$ be a measure-preserving action.*

1. *$A \subseteq M = L(\Gamma \circlearrowleft X)$ is **maximal abelian** in M (i.e. $A' \cap M = A$) if and only if $\Gamma \circlearrowleft X$ is **essentially free** (i.e. $\mu(\{t \in X : gt = t\}) = 0$ for all $g \neq e$).*
2. *If $\Gamma \circlearrowleft X$ is essentially free, then M is a factor if and only if σ is **ergodic** (if $a \in L^\infty(X)$ and $\sigma_g(a) = a$ for all g, then $a \in \mathbb{C}1$). So if Γ is infinite, then M is a II_1 factor.*